

A Multidimensional Exponential Utility Indifference Pricing Model with Applications to Counterparty Risk

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Abstract

This paper considers exponential utility indifference pricing for a multidimensional non-traded assets model and provides two approximations for the utility indifference price: a linear approximation by Picard iteration and a semigroup approximation by splitting techniques. The key tool is the probabilistic representation for the utility indifference price by the solution of fully coupled linear forward-backward stochastic differential equations. We apply our methodology to study the counterparty risk of derivatives in incomplete markets.

Keywords: utility indifference pricing, quadratic BSDE, FBSDE, semilinear PDE, splitting method, counterparty risk of derivatives.

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1 Introduction

The purpose of this paper is to consider *exponential* utility indifference pricing in a multidimensional non-traded assets setting, which is motivated by our study of counterparty risk of derivatives in incomplete markets. Our interest is in pricing and hedging derivatives written on assets which are not traded. The market is incomplete as the risks arising from having exposure to non-traded assets cannot be fully hedged. We take a utility indifference approach whereby the utility indifference price for derivative is the cash amount the investor is willing to pay such that she is no worse off in expected utility terms than she would have been without the derivative.

There has been considerable research in the area of *exponential* utility indifference valuation, but despite the interest in this pricing and hedging approach, there have been relatively few explicit formulas derived. The well known *one dimensional non-traded assets model* is an exception and in a Markovian framework with a derivative written on a single non-traded asset, and partial hedging in a financial asset, Henderson and Hobson [21], Henderson [18], and Musiela and Zariphopoulou [44] use the Cole-Hopf transformation (or *distortion power*) to linearize the non-linear PDE for the value function. This trick results in an explicit formula for the exponential utility indifference price. Subsequent generalizations of the model from Tehranchi [49], Frei and Schweizer [16] and [17] have shown the exponential utility indifference value can still be written in a closed-form expression similar to that known for the Brownian setting, although the structure of the formula can be much less explicit.

As soon as one of the assumptions made in the one dimensional non-traded asset breaks down, explicit formulas are no longer available. For example, if the option payoff depends also on the traded asset, Sircar and Zariphopoulou [47] develop bounds and asymptotic expansions for the exponential utility indifference price. In an energy context, we may be interested in partially observed models and need filtering techniques to numerically compute expectations (see Carmona and Ludkovski [11] and Chapter 7 of [10]). If the utility function is not exponential, Henderson [18] and Kramkov and Sirbu [32] developed expansions in small quantity for utility indifference prices under power utility.

In this paper, we study the exponential utility indifference price in a multidimensional setting with the aim of developing a pricing methodology. The main economic motivation for us to develop the multidimensional framework is to consider the counterparty default risk of options traded in over the counter (OTC) markets, often called *vulnerable options*. The recent credit crisis has brought to the forefront the importance of counterparty default risk as there were numerous high profile defaults leading to counterparty losses. In response, there have been many recent studies (for example, Bielecki, Crepey, Jeanblanc and Zargari [6] and Brigo and Chourdakis [8]) addressing in particular the counterparty risk of CDS. In contrast, there is relatively little recent work on counterparty risk for other derivatives, despite OTC options being a sizable fraction of the OTC derivatives market.¹ The option holder faces both price risk arising from the fluctuation of the assets underlying her option and counterparty default risk that the option writer does not honor her obligations. Default occurs when the

¹In fact, OTC options comprised about 10% of the \$600 trillion (in terms of notional amounts) OTC derivatives market at the end of June 2010 whilst the CDS market was about half as large at around \$30 trillion.

assets of the counterparty are below its liabilities at maturity (following the structural approach of Merton). In our setting, the assets of the counterparty and the asset underlying the option are non-traded and thus a multidimensional non-traded assets model naturally arises. A second potential area of application is to residual or basis risks arising when the asset(s) used for hedging differ from the assets underlying the contract in question (see Davis [13]).

Our first contribution is to use the solution of fully coupled linear forward-backward stochastic differential equation to give a probabilistic representation for the exponential utility indifference price. Since the associated equations are linear (but with a loop), we call it a *pseudo linear pricing rule*. It is well known that the utility indifference price can be written as a nonlinear expectation of payoff under the original physical measure, and the nonlinear expectation is often specified by a backward stochastic differential equation with quadratic growth (*quadratic BSDE* for short). Several authors derive quadratic BSDE representations of exponential utility indifference values in models of varying generality - see Mania and Schweizer [39], Ankirchner et al [1], Becherer [5], and Frei and Schweizer [16] [17] among others. In contrast to representing the utility indifference price by nonlinear expectation, we represent it as the linear expectation of payoff under some equivalent pricing measure. The pricing measure we choose may depend on the utility indifference price itself, so the pricing mechanism is a loop. What makes this transfer possible is the nonlinear Girsanov's transformation in the sense that the change of probability measure involves the solution of the equation, and the *BMO* martingale property of the solution, both of which essentially follow from the comparison principle of quadratic BSDEs (see for example Hu et al [23]). This kind of *pseudo linear pricing rule* appeared in Proposition 11 of Mania and Schweizer [39], where they modeled the payoff as a general random variable. In contrast, as we specify the dynamics of the underlying assets and the payoff structure, a fully coupled linear FBSDE appears naturally. Since the associated equations are linear (but with a loop), Picard iteration can be employed to approximate the solution and the corresponding utility indifference price. This can be regarded as a first application of the *pseudo linear pricing rule*.

Our second contribution is to develop a semigroup approximation for the pricing PDE by the splitting method, where we specialize to a Markovian setting and derive the semilinear pricing PDE with quadratic gradients satisfied by the utility indifference price. In our multidimensional setting, the Cole-Hopf transformation (as in the one dimensional model) can not be applied directly since the coefficients of the quadratic gradient terms do not match. Motivated by the idea of the splitting method (*or fractional step, prediction and correction*) in numerical analysis, we split the pricing equation into two semilinear PDEs with quadratic gradients such that the Cole-Hopf transformation can be applied to linearize both of them. Splitting methods have been used to construct numerical schemes for PDEs arising in mathematical finance - see the review of Barles [2], and Tourin [50] with the references therein. Recently, Nadtochiy and Zariphopoulou [45] applied splitting to the *marginal* Hamilton-Jacobi-Bellman (HJB) equation arising from optimal investment in a two-factor stochastic volatility model with general utility functions. They show their scheme converges to the unique viscosity solution of the limiting equation.

The idea of splitting in our setting is as follows. The local time derivative of the pricing PDE depends on the sum of semigroup operators corresponding to

the different factors. These semigroups usually are of different nature. For each sub-problem corresponding to each semigroup there might be an effective way providing solutions. For the sum of these semigroups, however, we cannot find an accurate method. The application of splitting method means that instead of the sum, we treat the semigroup operators separately. We prove that when the mesh of the time partition goes to zero, the approximated price will converge to the utility indifference price, relying on the *monotone scheme* argument by Barles and Souganidis [4]: any *monotone, stable and consistent* numerical scheme converges (to the correct solution) provided there exists a comparison principle for the limiting equations. The difficult part of applying the *monotone scheme* to our problem is the verification of consistency. This is overcome by the *pseudo linear pricing rule* for the utility indifference price. In contrast to the nonlinear expectation, where the Dominated Convergence Theorem may not hold, the representation is linear, so Dominated Convergence Theorem can be employed to verify the commute of limiting processes. This can be regarded as a second application of the *pseudo linear pricing rule*.

Our third contribution is to apply the splitting method to compute prices of derivatives on a non-traded asset and where the derivative holder is subject to non-traded counterparty default risk. In contrast to the complete market Black Scholes style formulas obtained by Johnson and Stulz [28], Klein [29] and Klein and Inglis [30], we show the significant impact that non-tradeable risks have on the valuation of vulnerable options and the role played by partial hedging.

The paper is organized as follows: In Section 2, we propose our multidimensional non-traded assets model, and present our probabilistic representation for the utility indifference price. Based on such representation, we give the Picard approximation for the price. In Section 3, we specialize to a Markovian setting, and apply the splitting method to give a semigroup approximation for the utility indifference pricing PDE. We further apply the scheme to the counterparty risk of derivatives in incomplete markets in Section 4.

2 Pseudo Linear Pricing Rule for Utility Indifference Valuation

Let $\mathcal{W} = (W^1, \dots, W^d)$ be a d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ satisfying the *usual conditions*, where \mathcal{F}_t is the augmented σ -algebra generated by $(\mathcal{W}_u : 0 \leq u \leq t)$. The market consists of a traded financial index P , whose price process is given by

$$\frac{dP_t}{P_t} = \mu_t^P dt + \langle \sigma_t^P, d\mathcal{W}_t \rangle, \quad (2.1)$$

and a set of observable but non-traded assets $\mathcal{S} = (S^1, \dots, S^n)$, whose price processes are given by

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \langle \sigma_t^i, d\mathcal{W}_t \rangle \quad (2.2)$$

for $i = 1, \dots, n$. $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^d with its Euclidean norm $\|\cdot\|$. We have $\mu_t^P, \mu_t^i \in \mathbb{R}$, $\sigma_t^P = (\sigma_t^{P1}, \dots, \sigma_t^{Pd}) \in \mathbb{R}^d$ and $\sigma_t^i = (\sigma_t^{i1}, \dots, \sigma_t^{id}) \in \mathbb{R}^d$. There is also a risk-free bond or bank account with price $B_t = 1$ for $t \geq 0$.

This is equivalent to working in terms of discounted units and is without loss of generality.

Our interest will be in pricing and hedging (path-dependent) contingent claims written on the non-traded assets \mathcal{S} . Specifically, we are concerned with contracts with payoff at maturity T of $g(\mathcal{S})$ which may depend on the whole path of \mathcal{S} . We impose the following assumptions, which will hold throughout:

- **Assumption (A1):** All the coefficients are \mathcal{F}_t -adapted and uniformly bounded in (t, ω) .
- **Assumption (A2):** The volatility for the financial index P is uniformly elliptic: $\|\sigma_t^P(\omega)\| \geq \epsilon > 0$ for all (t, ω) .
- **Assumption (A3):** The payoff g is a positive bounded functional.

The non-tradability of assets \mathcal{S} could be applicable in many situations - it might be that these assets are (i) not traded at all, or (ii) that they are traded illiquidly, or (iii) that they are in fact liquidly traded but the investor concerned is not permitted to trade them for some reason. Our main application is to the counterparty risk of derivatives where the payoff depends upon both the value of the counterparty's assets and the asset underlying the derivative itself. A second potential area of application is to residual or basis risks arising when the asset(s) used for hedging differ from the assets underlying the contract in question (see [13]). Typically this arises when the assets underlying the derivative are illiquidly traded (case (ii) above) and standardized futures contracts are used instead. Contracts may involve several assets, for example, a spread option with payoff $(K - S_T^1 - S_T^2)^+$ or a basket option with payoff $(K - S_T^1 - \dots - S_T^n)^+$. Such contracts frequently arise in applications to commodity, energy, and weather derivatives. Finally, a one dimensional example of the situation in (iii) is that of employee stock options (see Henderson [19]).

Our approach is to consider the utility indifference valuation for such contingent claims. For this we need to consider the optimization problem for the investor both with and without the option. The investor has initial wealth $x \in \mathbb{R}$, and is able to trade the financial index with price P_t (and riskless bond with price 1). This will enable the investor to partially hedge the risks she is exposed to via her position in the claim. Depending on the context, the financial index may be a stock, commodity or currency index, for example.

The holder of the option has an exponential utility function with respect to her terminal wealth:

$$U_T(x) = -e^{-\gamma x} \quad \text{for } \gamma \geq 0.$$

The investor holds λ units of the claim, whose price is denoted as \mathfrak{C}_0^λ and is to be determined², and invests her remaining wealth $x - \mathfrak{C}_0^\lambda$ in the financial index P . The investor will follow an admissible trading strategy:

$$\pi \in \mathcal{A}_{ad}[0, T] = \left\{ \pi : \pi \text{ is } \mathcal{F}_t\text{-adapted, and } \sup_{\tau} E^{\mathbf{P}} \left[\int_{\tau}^T |\pi_t|^2 dt \middle| \mathcal{F}_{\tau} \right] < \infty \right. \\ \left. \text{for any } \mathcal{F}_t\text{-stopping time } \tau \in [0, T]. \right\}$$

²For \mathfrak{C}_0^λ we use the superscript λ to emphasize the dependence of the price on the number of units held, and use the subscript 0 to denote the price at time 0. If one unit is held, we simply write \mathfrak{C}_0 rather than \mathfrak{C}_0^1 .

which results in the wealth:

$$X_t^{x-\mathfrak{C}_0^\lambda}(\pi) = x - \mathfrak{C}_0^\lambda + \int_0^t \pi_s (\mu_s^P ds + \langle \sigma_s^P, d\mathcal{W}_s \rangle). \quad (2.3)$$

The integrability condition for the trading strategies is essentially the *BMO* martingale property for $\int_0^\cdot \pi_s d\mathcal{W}_s$. However this condition is not restrictive if we only want to price and hedge contingent claims with bounded payoff, as the corresponding optimal trading strategy will satisfy this condition anyway.

The investor will optimize over such strategies to choose an optimal $\bar{\pi}^*$ by maximizing her expected terminal utility:

$$\sup_{\pi \in \mathcal{A}_{ad}[0,T]} E^{\mathbf{P}} \left[-e^{-\gamma \left(X_T^{x-\mathfrak{C}_0^\lambda}(\pi) + \lambda g(\mathcal{S}_\cdot) \right)} \right]. \quad (2.4)$$

To define the utility indifference price for the option, we also need to consider the optimization problem for the investor without the option. This involves the investor investing only in the financial index itself. Her wealth equation is the same as (2.3) but starts from initial wealth x and she will choose an optimal π^* by maximizing:

$$\sup_{\pi \in \mathcal{A}_{ad}[0,T]} E^{\mathbf{P}} \left[-e^{-\gamma X_T^x(\pi)} \right]. \quad (2.5)$$

We note that (2.5) is a special case of (2.4) with $\lambda = 0$.

The utility indifference price for option is the cash amount that the investor is willing to pay such that she is no worse off in expected utility terms than she would have been without the option. For a general overview of utility indifference pricing, we refer to the recent monograph edited by Carmona [10] and especially the survey article by Henderson and Hobson [22] therein.

Definition 2.1 (*Utility indifference valuation of option and hedge*)

The utility indifference price \mathfrak{C}_0^λ of λ units of the derivative with payoff $g(\mathcal{S}_\cdot)$ is defined by the solution to

$$\sup_{\pi \in \mathcal{A}_{ad}[0,T]} E^{\mathbf{P}} \left[-e^{-\gamma \left(X_T^{x-\mathfrak{C}_0^\lambda}(\pi) + \lambda g(\mathcal{S}_\cdot) \right)} \right] = \sup_{\pi \in \mathcal{A}_{ad}[0,T]} E^{\mathbf{P}} \left[-e^{-\gamma X_T^x(\pi)} \right]. \quad (2.6)$$

The hedging strategy for λ units of the derivative is defined by the difference in the optimal trading strategies $\bar{\pi}^* - \pi^*$.

One of main features of the utility indifference price is nonlinearity, i.e. $\mathfrak{C}_0^\lambda \neq \lambda \mathfrak{C}_0$. Given the assumption of exponential utility, the price of the option and corresponding hedging strategy can be represented by the solution of a quadratic BSDE.

Lemma 2.2 Suppose that Assumptions (A1) (A2) and (A3) are satisfied. Let (Y, \mathcal{Z}) be the unique solution of the quadratic BSDE:

$$Y_t = \lambda g(\mathcal{S}_\cdot) + \int_t^T f(s, \mathcal{Z}_s) ds - \int_t^T \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle, \quad (2.7)$$

where the driver $f(s, \mathcal{Z}_s)$ is given by

$$-\frac{\gamma}{2} \|\mathcal{Z}_t\|^2 + \frac{\gamma}{2\|\sigma_t^P\|^2} \left| \langle \sigma_t^P, \mathcal{Z}_t \rangle - \frac{\mu_t^P}{\gamma} \right|^2 - \frac{1}{2\gamma} \frac{|\mu_t^P|^2}{\|\sigma_t^P\|^2}.$$

The unique solution of BSDE (2.7) defines a nonlinear expectation $Y_t = \mathcal{E}^{\mathbf{P}}[\lambda g(\mathcal{S}) | \mathcal{F}_t]$. Then the utility indifference price \mathfrak{C}_0^λ is given by

$$\mathfrak{C}_0^\lambda = \mathcal{E}^{\mathbf{P}}[\lambda g(\mathcal{S})], \quad (2.8)$$

and the hedging strategy for λ units of the option is given by

$$-\frac{\langle \sigma_t^P, \mathcal{Z}_t \rangle}{\|\sigma_t^P\|^2}.$$

The above type of quadratic BSDE (2.7) can be derived by standard Martingale Optimality Principle (see, for example, Theorem 7 of Hu et al [23] and Section 3 of Ankirchner et al [1] in a Brownian motion setting, and Theorem 13 of Mania and Schweizer [39] and Section 2.1 of Morlais [42] in a general semimartingale setting). In the Appendix, we provide a different proof for Lemma 2.2 where we do not use Martingale Optimality Principle. Instead we consider the problem from risk-sensitive control prospective, and employ Girsanov's transformation and comparison principle for quadratic BSDEs to derive (2.7). We should also mention that considering utility indifference pricing from risk-sensitive control prospective seems to be new.

On the other hand, the well-posedness of quadratic BSDE (2.7) is guaranteed by Kobylanski [31]. Indeed, by Assumption (A3) the terminal data $\lambda g(\cdot)$ is uniformly bounded. The driver $f(t, \mathbf{z})$ for $\mathbf{z} = (z^1, \dots, z^d) \in \mathbb{R}^d$ is continuous in \mathbf{z} , and moreover, by Assumptions (A1) and (A2) on the coefficients, $f(t, \mathbf{z})$ satisfies

$$|f(t, \mathbf{z})| \leq C(1 + \|\mathbf{z}\|^2),$$

and

$$|\nabla_{\mathbf{z}} f(t, \mathbf{z})| \leq C(1 + \|\mathbf{z}\|), \quad \text{for } t \in [0, T], \text{ a.s..}$$

Hence, by Kobylanski [31], there exists a unique solution (Y, \mathcal{Z}) to BSDE (2.7).

The probabilistic representation formula (2.8) means the utility indifference price can be written as a *nonlinear* expectation of payoff under the original physical measure. However, this presentation is sometimes not convenient. First, the driver $f(t, \mathbf{z})$ for $\mathbf{z} = (z^1, \dots, z^d) \in \mathbb{R}^d$ is quadratic in \mathbf{z} , which is in particular not Lipschitz continuous. Secondly, some nice properties for linear expectation such as Dominated Convergence Theorem may not hold anymore for nonlinear expectation. Two natural questions to ask are

- Can the utility indifference price be written as a *linear* expectation of the payoff under some equivalent pricing measure? i.e.

$$\mathfrak{C}_0^\lambda = E^{\mathbf{Q}}[\lambda g(\mathcal{S})]. \quad (2.9)$$

- Is the above probabilistic representation formula (2.9) useful?

In the remainder of the paper, we shall show both of these questions have positive answers. We first try to find the pricing measure \mathbf{Q} in (2.9). Based on the solution (Y, \mathcal{Z}) to BSDE (2.7), we introduce the Doléans-Dade exponential in the following lemma:

Lemma 2.3 *The Doléans-Dade exponential $\mathcal{E}(N)$ defined by*

$$\mathcal{E}(N) = \exp \left\{ N - \frac{1}{2} [N, N] \right\},$$

where

$$N_t = - \int_0^t \frac{\gamma}{2} \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle + \int_0^t \frac{\gamma}{2 \|\sigma_s^P\|^2} \left\{ \langle \sigma_s^P, \mathcal{Z}_s \rangle - \frac{2\mu_s^P}{\gamma} \right\} \langle \sigma_s^P, d\mathcal{W}_s \rangle,$$

for $t \in [0, T]$ is a uniformly integrable martingale.

Proof. We first note that the solution Y of BSDE (2.7) is uniformly bounded and the process $\int_0^\cdot \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle$ is a \mathbf{P} -BMO martingale³, whose proof can be found, for example, in Lemma 12 of Hu et al [23].

Next, we verify that the Doléans-Dade exponential $\mathcal{E}(N)$ is a uniformly integrable martingale. Indeed, for any \mathcal{F}_t -stopping time $\tau \in [0, T]$,

$$\begin{aligned} & \sup_{\tau} E^{\mathbf{P}} [|N_T - N_{\tau}|^2 | \mathcal{F}_{\tau}] \\ &= \sup_{\tau} E^{\mathbf{P}} \left[\left| \int_{\tau}^T \frac{\gamma}{2} \langle \mathcal{Z}_t, d\mathcal{W}_t \rangle - \frac{\gamma}{2 \|\sigma_t^P\|^2} \left\{ \langle \sigma_t^P, \mathcal{Z}_t \rangle - \frac{2\mu_t^P}{\gamma} \right\} \langle \sigma_t^P, d\mathcal{W}_t \rangle \right|^2 | \mathcal{F}_{\tau} \right] \\ &\leq C \sup_{\tau} E^{\mathbf{P}} \left[\int_{\tau}^T \|\mathcal{Z}_t\|^2 dt + \int_{\tau}^T C dt | \mathcal{F}_{\tau} \right]. \end{aligned} \quad (2.10)$$

Since $\int_0^\cdot \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle$ is a \mathbf{P} -BMO martingale, i.e.

$$\sup_{\tau} E^{\mathbf{P}} \left[\left| \int_{\tau}^T \langle \mathcal{Z}_t, d\mathcal{W}_t \rangle \right|^2 | \mathcal{F}_{\tau} \right] = \sup_{\tau} E^{\mathbf{P}} \left[\int_{\tau}^T \|\mathcal{Z}_t\|^2 dt | \mathcal{F}_{\tau} \right] < \infty$$

for any \mathcal{F}_t -stopping time $\tau \in [0, T]$, the first term in the RHS of (2.10) is uniformly bounded. Clearly, the second term is bounded by $C^2 T$. Hence N is a \mathbf{P} -BMO martingale, and the Doléans-Dade exponential $\mathcal{E}(N)$ is uniformly integrable. ■

Since the Doléans-Dade exponential $\mathcal{E}(N)$ is uniformly integrable, we can change the probability measure from \mathbf{P} to \mathbf{Q} by $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(N)$. Under the measure \mathbf{Q} , BSDE (2.7) reduces to

$$\begin{aligned} Y_t &= \lambda g(\mathcal{S}_.) + \int_t^T f(s, \mathcal{Z}_s) ds - \int_t^T \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle \\ &= \lambda g(\mathcal{S}_.) - \int_t^T \langle \mathcal{Z}_s, d\mathcal{B}_s \rangle, \end{aligned}$$

where $\mathcal{B} = (B^1, \dots, B^d)$ defined by $\mathcal{B} = \mathcal{W} - [\mathcal{W}, N]$ is the Brownian motion under \mathbf{Q} by Girsanov's theorem. In other words, we have derived

$$Y_t = E^{\mathbf{Q}} [\lambda g(\mathcal{S}_.) | \mathcal{F}_t].$$

³We recall a continuous martingale M with $E^{\mathbf{P}} [M, M]_T < \infty$ is called a \mathbf{P} -BMO martingale if

$$\sup_{\tau} E^{\mathbf{P}} [|M_T - M_{\tau}|^2 | \mathcal{F}_{\tau}] < \infty$$

for any \mathcal{F}_t -stopping time $\tau \in [0, T]$. If M is a \mathbf{P} -BMO martingale, its Doléans-Dade exponential $\mathcal{E}(M)$ is in Doob's class \mathcal{D} , and therefore uniformly integrable.

Theorem 2.4 (*Pseudo linear pricing rule for utility indifference valuation*)

Suppose that Assumptions (A1) (A2) and (A3) are satisfied. Then there exists a pricing measure \mathbf{Q} defined by $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(N)$, where the Doléans-Dade exponential $\mathcal{E}(N)$ is defined in Lemma 2.3, such that the utility indifference price \mathfrak{C}_0^λ can be written as the linear expectation of the payoff g under \mathbf{Q} :

$$\mathfrak{C}_0^\lambda = E^{\mathbf{Q}}[\lambda g(\mathcal{S})].$$

Proof. The main steps are already shown above. What is left is that we only need to show the martingale representation part \mathcal{Z} is invariant under the equivalent change of probability measure. Actually, this follows by the uniqueness of the special semimartingale decomposition, whose proof can be found in Lemma 7 of Liang et al [37]. ■

The probabilistic representation of the utility indifference price (2.9) appears to be “linear” at first glance. Notwithstanding, the nonlinearity is hidden in the pricing measure \mathbf{Q} , which depends on the price \mathfrak{C}_0^λ through the Doléans-Dade exponential $\mathcal{E}(N)$. Indeed, if the unit indifference price is given by $\mathfrak{C}_0 = E^{\mathbf{Q}'}[g(\mathcal{S})]$, where \mathbf{Q}' is defined by the Doléans-Dade exponential which depends on the price \mathfrak{C}_0 , then,

$$\mathfrak{C}_0^\lambda = E^{\mathbf{Q}}[\lambda g(\mathcal{S})] = \lambda E^{\mathbf{Q}}[g(\mathcal{S})] \neq \lambda E^{\mathbf{Q}'}[g(\mathcal{S})] = \lambda \mathfrak{C}_0.$$

We are not the first to discover such a *pseudo linear pricing rule* for the exponential utility indifference price. A similar probabilistic representation to Theorem 2.4 is given in Proposition 11 of Mania and Schweizer [39], where their payoff is a general random variable. In contrast, since we specify the dynamics of the underlying assets and the payoff structure, a fully coupled linear FBSDE appears naturally. Indeed, if we write down the equations of the non-traded assets \mathcal{S} under the measure \mathbf{Q} :

$$\frac{dS_t^i}{S_t^i} = \left\{ \mu_t^i - \frac{\gamma}{2} \langle \sigma_t^i, \mathcal{Z}_t \rangle + \frac{\gamma \langle \sigma_t^i, \sigma_t^P \rangle}{2 \|\sigma_t^P\|^2} \left(\langle \sigma_t^P, \mathcal{Z}_t \rangle - \frac{2\mu_t^P}{\gamma} \right) \right\} dt + \langle \sigma_t^i, d\mathcal{B}_t \rangle,$$

together with the dynamic of the utility indifference price \mathfrak{C}_0^λ under the measure \mathbf{Q} :

$$Y_t = \lambda g(\mathcal{S}) - \int_t^T \langle \mathcal{Z}_s, d\mathcal{B}_s \rangle,$$

we derive a fully coupled linear FBSDE. Since the associated equations are linear (but with a loop), Picard iteration can be employed to approximate the solution and the corresponding utility indifference price, which results in the following approximation, whose proof is postponed to the Appendix.

Theorem 2.5 (*Linear approximation for utility indifference price*)

Suppose that Assumptions (A1) (A2) and (A3) are satisfied, and moreover, for either T or K small enough, the payoff functional satisfies

$$|g(\mathcal{S}) - g(\bar{\mathcal{S}})| \leq K \sup_{t \in [0, T]} \sum_{i=1}^n |\ln S_t^i - \ln \bar{S}_t^i|.$$

Let $\mathcal{B} = (B^1, \dots, B^d)$ be the Brownian motion under a given probability measure \mathbf{Q} . Define the following sequence $\{\mathcal{S}^m = (S^{m,1}, \dots, S^{m,n})\}_{m \geq 0}$ iteratively: $\mathcal{S}^0 = (s, \dots, s)$, and

$$\frac{dS_t^{m+1,i}}{S_t^{m+1,i}} = \left\{ \mu_t^i - \frac{\gamma}{2} \langle \sigma_t^i, \mathcal{Z}_t^m \rangle + \frac{\gamma \langle \sigma_t^i, \sigma_t^P \rangle}{2 \|\sigma_t^P\|^2} \left(\langle \sigma_t^P, \mathcal{Z}_t^m \rangle - \frac{2\mu_t^P}{\gamma} \right) \right\} dt + \langle \sigma_t^i, d\mathcal{B}_t \rangle,$$

where $\mathcal{Z}^m = (Z^{m,1}, \dots, Z^{m,d})$ is the martingale representation of $\lambda g(\mathcal{S}^m)$:

$$\int_t^T \langle \mathcal{Z}_s^m, d\mathcal{B}_s \rangle = \lambda g(\mathcal{S}^m) - E^{\mathbf{Q}}[\lambda g(\mathcal{S}^m) | \mathcal{F}_t].$$

Then the utility indifference price \mathcal{C}_0^λ is approximated by

$$\mathfrak{C}_0^{m,\lambda} = E^{\mathbf{Q}}[\lambda g(\mathcal{S}^m)],$$

and

$$\lim_{m \rightarrow \infty} \mathfrak{C}_0^{m,\lambda} = \mathfrak{C}_0^\lambda.$$

To summarize, Theorem 2.4 and Lemma 2.2 provide two alternative ways to represent the utility indifference price:

$$\mathfrak{C}_0^\lambda = E^{\mathbf{Q}}[\lambda g(\mathcal{S})] = \mathcal{E}^{\mathbf{P}}[\lambda g(\mathcal{S})].$$

One application of our representation in Theorem 2.4 was illustrated in Theorem 2.5. A second application will be given in Section 3 where we shall utilize the representation to prove the convergence of semigroup approximation for the utility indifference price by splitting method in Theorem 3.8.

3 The Markovian Case and the Splitting Method

In this section, we specialize to a Markovian setting by assuming that all the coefficients are constant, denoted by $\mu_P, \sigma_P, \mu_i, \sigma_i$ etc., and the payoff g is a function rather than a functional.

- **Assumption (A1)'**: All the coefficients are nonzero constants.
- **Assumption (A3)'**: The payoff g is a positive bounded function.

We consider a one factor model where the prices of the traded financial index P and each of the non-traded assets S^i are driven by one common market Brownian motion, as well as an independent Brownian motion, representing the idiosyncratic risk of each asset. In other words, $d = n + 2$, $\sigma_t^{pj} = 0$ if $j \neq n + 1$ and $n + 2$, and $\sigma_t^{ij} = 0$ if $i \neq j$ and $j \neq n + 1$. The price processes become

$$\frac{dP_t}{P_t} = \mu_P dt + \bar{\sigma}_P dW_t^{n+1} + \sigma_P dW_t^{n+2}, \quad (3.1)$$

and

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sigma_i dW_t^i + \bar{\sigma}_i dW_t^{n+1}. \quad (3.2)$$

To simplify the notation, we set $\bar{\sigma}_P = \sigma_{P,n+1}$, $\sigma_P = \sigma_{P,n+2}$, $\sigma_i = \sigma_{ii}$, and $\bar{\sigma}_i = \sigma_{i,n+1}$.

The price of each non-traded asset S^i reflects exposure to the traded or market risk W^{n+1} through volatility $\bar{\sigma}_i$ and idiosyncratic risk W^i through idiosyncratic volatility σ_i . We define the following parameters for the financial index P :

$$\begin{aligned}\theta^P &= \frac{(\mu_P)^2}{(\sigma_P)^2}; & \bar{\theta}^P &= \frac{(\mu_P)^2}{(\sigma_P)^2 + (\bar{\sigma}_P)^2}; \\ \vartheta^P &= \frac{\mu_P \bar{\sigma}_P}{(\sigma_P)^2}; & \bar{\vartheta}^P &= \frac{\mu_P \bar{\sigma}_P}{(\sigma_P)^2 + (\bar{\sigma}_P)^2}; \\ \kappa^P &= \frac{(\bar{\sigma}_P)^2}{(\sigma_P)^2}; & \bar{\kappa}^P &= \frac{(\bar{\sigma}_P)^2}{(\sigma_P)^2 + (\bar{\sigma}_P)^2}.\end{aligned}$$

We first characterize the utility indifference price by the solution of the following semilinear PDE with quadratic gradients.

Lemma 3.1 *Suppose that Assumptions (A1)' (A2) and (A3)' are satisfied, and moreover, the payoff function $g(\cdot)$ is Lipschitz continuous. Then the utility indifference price of λ units of the option with payoff $g(S_T)$ is given by $\mathfrak{C}^\lambda(\cdot, 0)$, where \mathfrak{C}^λ is the classical solution of the PDE:*

$$\begin{cases} \partial_t \mathfrak{C}^\lambda(\mathbf{s}, t) + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 s_i^2 \partial_{s_i s_i} \mathfrak{C}^\lambda + \frac{1}{2} \sum_{i,j=1}^n \bar{\sigma}_i \bar{\sigma}_j s_i s_j \partial_{s_i s_j} \mathfrak{C}^\lambda \\ + \sum_{i=1}^n (\mu_i - \bar{\vartheta}^P \bar{\sigma}_i) s_i \partial_{s_i} \mathfrak{C}^\lambda - \frac{\gamma}{2} \sum_{i=1}^n \sigma_i^2 s_i^2 (\partial_{s_i} \mathfrak{C}^\lambda)^2 \\ - \frac{\gamma}{2} \sum_{i,j=1}^n [(1 - \bar{\kappa}^P) \bar{\sigma}_i \bar{\sigma}_j] s_i s_j \partial_{s_i} \mathfrak{C}^\lambda \partial_{s_j} \mathfrak{C}^\lambda = 0, \\ \mathfrak{C}^\lambda(\cdot, T) = \lambda g(\cdot) \end{cases} \quad (3.3)$$

on the domain $(\mathbf{s}, t) \in \mathbb{R}_+^n \times [0, T]$, where $\mathbf{s} = (s_1, \dots, s_n)$, and the hedging strategy for λ units of the option is given by

$$- \frac{\bar{\kappa}^P}{\bar{\sigma}_P} \sum_{i=1}^n \bar{\sigma}_i s_i \partial_{s_i} \mathfrak{C}^\lambda. \quad (3.4)$$

Since the proof of Lemma 3.1 follows from the standard Dynamic Programming Principle, we omit it (for example see [18] in one dimensional setting). We only remark that the classical solution to (3.1) exists, and moreover, \mathfrak{C}^λ and its gradients are uniformly bounded (see, for example, Theorem 2.9 of Delarue [14] for the proof). We also note that the number of units λ only appears in the terminal condition. In the following, we present the case $\lambda = 1$, and the price is simply denoted by \mathfrak{C} .

In general, there is no explicit solution to PDE (3.3). However, if the traded financial index is independent of the non-traded assets, i.e. $\bar{\sigma}_P = 0$, the explicit solution is available by the Cole-Hopf transformation (see comments on the one-dimensional case after Proposition 3.6). Essentially, this means the financial index P is not useful as a hedging tool for the investor.

Corollary 3.2 Suppose that $\bar{\sigma}_P = 0$, and Assumptions (A1)' (A2) and (A3)' are satisfied, and moreover, $g(\cdot)$ is Lipschitz continuous. Then the utility indifference price of the option with payoff $g(\mathcal{S}_T)$ is given by

$$\mathfrak{C} = -\frac{1}{\gamma} \ln E^{\mathbf{P}} \left[e^{-\gamma g(\mathcal{S}_T)} \right]. \quad (3.5)$$

We now contrast the above to the situation if the market were complete. If the underlying assets $\mathcal{S} = (S^1, \dots, S^n)$ could be traded, the market would become complete, and the pricing and hedging of the contingent claim with payoff $g(\mathcal{S}_T)$ falls into the classical multidimensional Black-Scholes framework.

Corollary 3.3 Suppose that $\mathcal{S} = (S^1, \dots, S^n)$ are traded assets, and Assumptions (A1)' (A2) and (A3)' are satisfied, and moreover, $g(\cdot)$ is Lipschitz continuous. Then the price of the option with payoff $g(\mathcal{S}_T)$ is given by $\bar{\mathfrak{C}}$, where $\bar{\mathfrak{C}}$ is the classical solution of the PDE:

$$\begin{cases} \partial_t \bar{\mathfrak{C}}(\mathbf{s}, t) + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 s_i^2 \partial_{s_i s_i} \bar{\mathfrak{C}} + \frac{1}{2} \sum_{i,j=1}^n \bar{\sigma}_i \bar{\sigma}_j s_i s_j \partial_{s_i s_j} \bar{\mathfrak{C}} = 0, \\ \bar{\mathfrak{C}}(\cdot, T) = g(\cdot) \end{cases} \quad (3.6)$$

on the domain $(\mathbf{s}, t) \in \mathbb{R}_+^n \times [0, T]$.

Based on the pricing equation (3.3), we will present a number of properties of the option price. First notice that the pricing equation (3.3) has additional nonlinear terms of quadratic gradients relative to the complete market PDE in (3.6).

Our first property of the utility indifference price concerns risk aversion. Intuitively, the more risk averse the option holder is, the less she would be willing to pay for the option.

Proposition 3.4 If the risk aversion parameter γ increases, the unit utility indifference price \mathfrak{C} will decrease.

Proof. The proof is based on the comparison principle for PDE (3.3). Suppose $0 \leq \gamma_1 \leq \gamma_2$. The corresponding PDEs are denoted by PDE^{γ_1} and PDE^{γ_2} respectively, and their solutions are denoted by \mathfrak{C}^{γ_1} and \mathfrak{C}^{γ_2} respectively. Since the terms involving γ can be regrouped as

$$-\frac{\gamma}{2} \sum_{i=1}^n \sigma_i^2 s_i^2 (\partial_{s_i} \mathfrak{C})^2 - \frac{\gamma}{2} (1 - \bar{\kappa}^P) \left(\sum_{i=1}^n \bar{\sigma}_i s_i \partial_{s_i} \mathfrak{C} \right)^2$$

so all the coefficients of γ are less than zero, we have

$$PDE^{\gamma_1}(\mathfrak{C}^{\gamma_2}) \geq 0.$$

Hence \mathfrak{C}^{γ_2} is the subsolution to PDE^{γ_1} . On the other hand \mathfrak{C}^{γ_1} is the supersolution to PDE^{γ_1} , and $\mathfrak{C}^{\gamma_2}|_{t=T} = \mathfrak{C}^{\gamma_1}|_{t=T}$. By the comparison principle, we conclude $\mathfrak{C}^{\gamma_2} \leq \mathfrak{C}^{\gamma_1}$ on $\mathbb{R}_+^n \times [0, T]$. ■

Proposition 3.5 Assume that

$$\bar{\vartheta}^P = \mu_i / \bar{\sigma}_i \quad (3.7)$$

for $i = 1, \dots, n$. Then the unit utility indifference price \mathfrak{C} is decreasing in the idiosyncratic volatility of the traded asset σ_P^2 (or its proportion of total volatility, $1 - \bar{\kappa}^P$).

Proof. The proof is again based on the comparison principle for (3.3), and is similar to that of Proposition 3.4. ■

This tells us that the higher the idiosyncratic volatility σ_P^2 of the traded asset (or as a proportion of total volatility), the worse it is as a hedging instrument, and the lower the price one is willing to pay. Proposition 3.5 generalizes the monotonicity obtained in the one dimensional non-traded asset model (see for example, Henderson [19] and Frei and Schweizer [16] in a non-Markovian model with stochastic correlation).

The restriction (3.7) in fact corresponds to a relation between the Sharpe ratios of the non-traded assets S and the financial index P . Define the Sharpe ratio of S^i to be $SR_i = \mu_i / \sqrt{\sigma_i^2 + \bar{\sigma}_i^2}$ and similarly, the Sharpe ratio for the financial index P by $SR_P = \mu_P / \sqrt{\sigma_P^2 + \bar{\sigma}_P^2}$. Then (3.7) is equivalent to the relation

$$SR_i = \left(\frac{\bar{\sigma}_i \bar{\sigma}_P}{\sqrt{\bar{\sigma}_i^2 + \sigma_i^2} \sqrt{\bar{\sigma}_P^2 + \sigma_P^2}} \right) SR_P = \rho_{iP} SR_P$$

where ρ_{iP} is the correlation between S^i and P . This corresponds to the relation we expect from the capital asset pricing model (CAPM) when assets are traded. Since not all assets are traded here, we would not necessarily expect (3.7) to hold.

The final result in this section concerns recovery of the complete market price given in Corollary 3.3.

Proposition 3.6 (*Asymptotic results*)

- (i) If $\bar{v}^P = \frac{\mu_i}{\bar{\sigma}_i}$ for $i = 1, \dots, n$, then the unit utility indifference price \mathfrak{C} will converge to the complete market price $\bar{\mathfrak{C}}$ as $\gamma \rightarrow 0$.
- (ii) If $\frac{\mu_P}{\bar{\sigma}_P} = \frac{\mu_i}{\bar{\sigma}_i}$ for $i = 1, \dots, n$, then as $\sigma_P, \sigma_i \rightarrow 0$ the unit utility indifference price \mathfrak{C} will converge to the complete market price $\bar{\mathfrak{C}}$ with $\sigma_P, \sigma_i = 0$.

Proof. We prove (i), whilst the proof of (ii) is similar. Under the assumptions in (i), when $\gamma \rightarrow 0$, by the Arzela-Ascoli compactness criterion, there exists a subsequence $\gamma_n \rightarrow 0$ such that the solutions of (3.3), denoted by \mathfrak{C}^{γ_n} , uniformly converge to $\bar{\mathfrak{C}}$ on any compact subset of $\mathbb{R}_+^n \times [0, T]$, where $\bar{\mathfrak{C}}$ satisfies (3.6). ■

Again, we see in Proposition 3.6 that the CAPM restrictions must apply to the Sharpe ratios. The intuition here is that when the idiosyncratic volatilities disappear, and when assets are traded, there cannot be a difference in using the financial index P or the assets themselves to hedge.

For a semilinear PDE with quadratic gradients like (3.3), it is not usually possible to obtain an explicit solution. A special case where an explicit solution does exist is the one dimensional version. Taking $n = 1$, $d = 2$ and $\sigma_{P1} = 0$ in (3.3) recovers the pricing PDE of [21], [18] and [44], which is solved by the Cole-Hopf transformation. However, this transformation does not apply directly to our multidimensional problem (3.3) because the coefficients of the quadratic gradient terms in (3.3) do not match. We note that even applying standard finite difference method to numerically solve the PDE (3.3) is troublesome due to the high dimension and nonlinearity. Instead, we will develop a splitting algorithm

which will enable us to take advantage again of the Cole-Hopf transformation to linearize the PDEs.

Splitting methods (*or fractional step, prediction and correction*) can be dated back to Marchuk [40] in the late 1960's (see also [41]). Application of splitting idea to nonlinear PDEs such as HJB equations is difficult mainly because of the verification of the convergence for the approximation scheme. This difficulty was overcome by Barles and Souganidis [4], who employed the idea of the viscosity solution and proved that any *monotone, stable and consistent* numerical scheme converges (to the correct solution) provided there exists a comparison principle for the limiting equation. Barles and Souganidis [4] pointed out that their result could be used to justify most standard splitting methods.

The idea of splitting in our setting is the following. The local time derivative of the pricing PDE (3.3) depends on the sum of semigroup operators (*or the associated infinitesimal operators*) corresponding to the different factors. These semigroups usually are of different nature. For each sub-problem corresponding to each semigroup there might be an effective way providing solutions. For the sum of these semigroups, however, we usually can not find an accurate method. Hence, application of splitting method means that instead of the sum, we treat the semigroup operators separately.

Of course, the tricky part is how to split the equation (*or how to group factors*) effectively. In the following, we separate the pricing PDE (3.3) into two pricing factors. Take $\lambda = 1$. We first make the log-transformation: $x_i = \ln s_i$, and define a new operator:

$$\frac{\partial}{\partial \eta} = \sum_{i=1}^n \bar{\sigma}_i \frac{\partial}{\partial x_i}. \quad (3.8)$$

Then (3.3) reduces to

$$\begin{aligned} \partial_t \mathfrak{C} + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 \partial_{x_i x_i} \mathfrak{C} + \sum_{i=1}^n A_i \partial_{x_i} \mathfrak{C} - \frac{\gamma}{2} \sum_{i=1}^n \sigma_i^2 (\partial_{x_i} \mathfrak{C})^2 \\ + \frac{1}{2} \partial_{\eta\eta} \mathfrak{C} - \frac{\gamma}{2} (1 - \bar{\kappa}^P) (\partial_{\eta} \mathfrak{C})^2 = 0, \end{aligned} \quad (3.9)$$

where

$$A_i = \mu_i - \frac{1}{2} (\sigma_i^2 + \bar{\sigma}_i^2) - \bar{\vartheta}^P \bar{\sigma}_i. \quad (3.10)$$

Define two operators:

$$\begin{aligned} \mathbf{L}^1 &= \frac{1}{2} \partial_{\eta\eta} - \frac{\gamma}{2} (1 - \bar{\kappa}^P) \partial_{\eta}^2 \\ \mathbf{L}^2 &= \frac{1}{2} \sum_{i=1}^n \sigma_i^2 \partial_{x_i x_i} + \sum_{i=1}^n A_i \partial_{x_i} - \frac{\gamma}{2} \sum_{i=1}^n \sigma_i^2 \partial_{x_i}^2. \end{aligned}$$

For any $0 \leq T_1 < T_2 \leq T$, and any smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^m)$ for $m = 1, n$, we define the following nonlinear backward semigroup operators $\mathbf{S}^i(T_1, T_2) : \mathcal{C}^\infty(\mathbb{R}^m) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m)$ by $\phi(\cdot) \mapsto \mathfrak{C}^i(\cdot, T_1)$ where

$$\partial_t \mathfrak{C}^i + \mathbf{L}^i \mathfrak{C}^i = 0; \quad \mathfrak{C}^i(\cdot, T_2) = \phi(\cdot)$$

on the domain $[T_1, T_2] \times \mathbb{R}^m$ for $i = 1, 2$.

We observe that one of the equations can be linearized by a Cole-Hopf transformation. Indeed, by letting $\bar{\mathfrak{C}}^1 = \exp(-\gamma(1 - \bar{\kappa}^P)\mathfrak{C}^1)$, then we have $\bar{\mathfrak{C}}^1$ satisfying

$$\partial_t \bar{\mathfrak{C}}^1 + \frac{1}{2} \partial_{\eta\eta} \bar{\mathfrak{C}}^1 = 0. \quad (3.11)$$

By letting $\bar{\mathfrak{C}}^2 = \exp(-\gamma\mathfrak{C}^2)$, then we have $\bar{\mathfrak{C}}^2$ satisfying

$$\partial_t \bar{\mathfrak{C}}^2 + \frac{1}{2} \sum_{i=1}^n \sigma_i^2 \partial_{x_i x_i} \bar{\mathfrak{C}}^2 + \sum_{i=1}^n A_i \partial_{x_i} \bar{\mathfrak{C}}^2 = 0. \quad (3.12)$$

Lemma 3.7 *The operators $\mathbf{S}^i(T_1, T_2)$ for $i = 1, 2$ have the following properties:*

- (i) *For any smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^m)$,*

$$\lim_{T_1 \uparrow T_2} \mathbf{S}^i(T_1, T_2) \phi = \phi$$

uniformly on any compact subset of \mathbb{R}^m .

- (ii) *For any $0 \leq T_1 < T_2 < T_3 \leq T$,*

$$\mathbf{S}^i(T_1, T_3) \phi = \mathbf{S}^i(T_1, T_2) \mathbf{S}^i(T_2, T_3) \phi.$$

- (iii)

$$\mathbf{S}^i(T_2, T_2) \phi = \phi.$$

((i) (ii) and (iii) ensure that $\mathbf{S}^i(T_1, T_2)$ is indeed a semigroup operator.)

- (iv) *If $\phi \geq \psi$ where ψ is another smooth function, then*

$$\mathbf{S}^i(T_1, T_2) \phi \geq \mathbf{S}^i(T_1, T_2) \psi.$$

- (v) *For any constant $k \in \mathbb{R}$,*

$$\mathbf{S}^i(T_1, T_2)(\phi(\cdot) + k) = \mathbf{S}^i(T_1, T_2)(\phi(\cdot)) + k.$$

- (vi)

$$\lim_{T_1 \uparrow T_2} \frac{\mathbf{S}^i(T_1, T_2) \phi - \phi}{T_2 - T_1} + \mathbf{L}^i \phi = 0$$

uniformly on any compact subset of \mathbb{R}^m .

Proof. (i)-(v) are immediate. We only prove (vi) in the following. We first prove the case $i = 1$. Note that $\mathbf{S}^1(T_1, T_2) = \mathbf{S}^1(0, T_2 - T_1)$, and it is enough to prove that for any $\phi \in \mathcal{C}^\infty(\mathbb{R})$,

$$\lim_{T_2 - T_1 \downarrow 0} \frac{\mathbf{S}^1(0, T_2 - T_1) \phi - \phi}{T_2 - T_1} = \mathbf{L}^1 \phi$$

uniformly on any compact subset of \mathbb{R} .

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ be a filtered probability space satisfying the usual conditions, on which supports Brownian motion W with $\mathbf{P}(W_0 = \eta) = 1$. By Itô's formula, we have

$$\begin{aligned} & \mathfrak{C}^1(t, W_t) \\ &= \mathfrak{C}^1(T_2 - T_1, W_{T_2 - T_1}) - \int_t^{T_2 - T_1} (\partial_t \mathfrak{C}^1 + \frac{1}{2} \partial_{\eta\eta} \mathfrak{C}^1)(s, W_s) ds - \int_t^{T_2 - T_1} \partial_\eta \mathfrak{C}^1(s, W_s) dW_s \\ &= \phi(W_{T_2 - T_1}) - \int_t^{T_2 - T_1} \frac{\gamma}{2} (1 - \bar{\kappa}^P) (\partial_\eta \mathfrak{C}^1)^2(s, W_s) ds - \int_t^{T_2 - T_1} \partial_\eta \mathfrak{C}^1(s, W_s) dW_s \end{aligned}$$

for $t \in [0, T_2 - T_1]$. We note that $\partial_\eta \mathfrak{C}^1(t, \eta)$ is uniformly bounded, and therefore,

$$N_t = - \int_0^t \frac{\gamma}{2} (1 - \bar{\kappa}^P) \partial_\eta \mathfrak{C}^1(s, W_s) dW_s, \quad \text{for } t \in [0, T_2 - T_1]$$

is a \mathbf{P} -BMO martingale, which implies that the Doléans-Dade exponential $\mathcal{E}(N)$ is uniformly integrable. Hence, we can define a new probability measure \mathbf{Q} by $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(N)$. By Girsanov's theorem,

$$B_t = W_t - [W, N]_t = W_t + \int_0^t \frac{\gamma}{2} (1 - \bar{\kappa}^P) \partial_\eta \mathfrak{C}^1(s, W_s) ds, \quad \text{for } t \in [0, T]$$

is Brownian motion under the probability measure \mathbf{Q} , and moreover,

$$\begin{aligned} \mathfrak{C}^1(t, W_t) &= \phi(W_{T_2 - T_1}) - \int_t^{T_2 - T_1} \partial_\eta \mathfrak{C}^1(s, W_s) dB_s \\ &= E^{\mathbf{Q}}[\phi(W_{T_2 - T_1}) | \mathcal{F}_t]. \end{aligned}$$

Therefore, by Itô's formula, we have

$$\begin{aligned} & \frac{\mathbf{S}^1(0, T_2 - T_1) \phi(\eta) - \phi(\eta)}{T_2 - T_1} \\ &= \frac{1}{T_2 - T_1} E^{\mathbf{Q}}[\phi(W_{T_2 - T_1})] - \phi(\eta) \\ &= \frac{1}{T_2 - T_1} E^{\mathbf{Q}} \left[\int_0^{T_2 - T_1} \partial_\eta \phi(W_s) dW_s + \int_0^{T_2 - T_1} \frac{1}{2} \partial_{\eta\eta} \phi(W_s) d\langle W \rangle_s \right] \\ &= \frac{1}{T_2 - T_1} E^{\mathbf{Q}} \left[\int_0^{T_2 - T_1} \partial_\eta \phi(W_s) dB_s - \int_0^{T_2 - T_1} \frac{\gamma}{2} (1 - \bar{\kappa}^P) \partial_\eta \phi(W_s) \partial_\eta \mathfrak{C}^1(s, W_s) ds \right. \\ & \quad \left. + \int_0^{T_2 - T_1} \frac{1}{2} \partial_{\eta\eta} \phi(W_s) ds \right] \\ &= \frac{1}{T_2 - T_1} E^{\mathbf{Q}} \left[\int_0^{T_2 - T_1} -\frac{\gamma}{2} (1 - \bar{\kappa}^P) \partial_\eta \phi(W_s) \partial_\eta \mathfrak{C}^1(s, W_s) + \frac{1}{2} \partial_{\eta\eta} \phi(W_s) ds \right]. \end{aligned}$$

Note that $\partial_\eta \mathfrak{C}^1(0, \eta) = \partial_\eta \phi(\eta)$ and by Dominated Convergence Theorem,

$$\lim_{T_2 - T_1 \downarrow 0} \frac{\mathbf{S}^1(0, T_2 - T_1) \phi(\eta) - \phi(\eta)}{T_2 - T_1} = -\frac{\gamma}{2} (1 - \bar{\kappa}^P) (\partial_\eta \phi)^2(\eta) + \frac{1}{2} \partial_{\eta\eta} \phi(\eta).$$

The proof for the case $i = 2$ is similar, so we only sketch its proof. We apply Itô's formula to $\mathfrak{C}^2(t, \mathcal{X}_t)$, where $\mathcal{X} = (X^1, \dots, X^n)$ is given by

$$X_t^i = x^i + \int_0^t A_i ds + \int_0^t \sigma_i dW_s^i.$$

Therefore, by changing the probability measure, we obtain

$$\begin{aligned} \mathfrak{C}^2(t, \mathcal{X}_t) &= \phi(\mathcal{X}_{T_2-T_1}) - \int_t^{T_2-T_1} \frac{\gamma}{2} \sum_{i=1}^n \sigma_i^2 (\partial_{x_i} \mathfrak{C}^2)^2(s, \mathcal{X}_s) ds \\ &\quad - \int_t^{T_2-T_1} \sum_{i=1}^n \sigma_i \partial_{x_i} \mathfrak{C}^2(s, \mathcal{X}_s) dW_s^i \\ &= E^{\mathbf{Q}}[\phi(\mathcal{X}_{T_2-T_1}) | \mathcal{F}_t], \end{aligned}$$

where \mathbf{Q} is defined by the Doléans-Dade exponential $\mathcal{E}(N)$ with

$$N_t = - \int_0^t \sum_{i=1}^n \frac{\gamma}{2} \sigma_i \partial_{x_i} \mathfrak{C}^2(s, \mathcal{X}_s) dW_s^i, \quad \text{for } t \in [0, T_2 - T_1].$$

The rest of the proof follows by the same argument as for the case $i = 1$. \blacksquare

Next we use semigroup operators $\mathbf{S}^1(T_1, T_2)$ and $\mathbf{S}^2(T_1, T_2)$ to give the semigroup approximation for the solution of PDE (3.9)(or PDE (3.3)), which is the main result of this section.

Theorem 3.8 (*Semigroup approximation for utility indifference price*)

Suppose that Assumptions (A1)' (A2) and (A3)' are satisfied, and moreover, the payoff function $g(\cdot)$ is Lipschitz continuous. Let $\pi : 0 = t_0 < t_1 < \dots < t_N = T$ be the partition of $[0, T]$ with mesh:

$$|\pi| := \max_{0 \leq i \leq N-1} |t_{i+1} - t_i|.$$

Then the unit utility indifference price of the derivative with payoff $g(\mathcal{S}_T)$, which is denoted by $\mathfrak{C}(\cdot, 0)$ is approximated by

$$\mathfrak{C}^\pi(\cdot, 0) = \prod_{i=0}^{N-1} \mathbf{S}^1(t_i, t_{i+1}) \mathbf{S}^2(t_i, t_{i+1}) g(\cdot),$$

and

$$\lim_{|\pi| \rightarrow 0} \mathfrak{C}^\pi(\cdot, 0) = \mathfrak{C}(\cdot, 0).$$

uniformly on any compact subset of \mathbb{R}^n .

Proof. The proof is based on the Barles-Souganidis monotone scheme [4], in which they proved that any *monotone, stable and consistent* numerical scheme converges (to the correct solution) provided there exists a comparison principle for the limiting equation.

In the following, we verify the above conditions. For any $0 \leq T_1 < T_2 \leq T$ and any smooth functions $\phi \geq \psi$, by (iv) in Lemma 3.7,

$$\mathbf{S}^1(T_1, T_2) \mathbf{S}^2(T_1, T_2) \phi \geq \mathbf{S}^1(T_1, T_2) \mathbf{S}^2(T_1, T_2) \psi,$$

so the scheme is monotone.

For any $k \in \mathbb{R}$, by (v) in Lemma 3.7,

$$\begin{aligned}\mathbf{S}^1(T_1, T_2)\mathbf{S}^2(T_1, T_2)(\phi + k) &= \mathbf{S}^1(T_1, T_2)(\mathbf{S}^2(T_1, T_2)\phi + k) \\ &= \mathbf{S}^1(T_1, T_2)\mathbf{S}^2(T_1, T_2)\phi + k\end{aligned}$$

so the scheme is stable.

Finally, we verify the scheme is consistent:

$$\begin{aligned}& \frac{\mathbf{S}^1(T_1, T_2)\mathbf{S}^2(T_1, T_2)\phi - \phi}{T_2 - T_1} + (\mathbf{L}^1 + \mathbf{L}^2)\phi \\ &= \frac{\mathbf{S}^1(T_1, T_2)\mathbf{S}^2(T_1, T_2)\phi - \mathbf{S}^2(T_1, T_2)\phi}{T_2 - T_1} + \mathbf{L}^1\mathbf{S}^2(T_1, T_2)\phi \\ & \quad + \frac{\mathbf{S}^2(T_1, T_2)\phi - \phi}{T_2 - T_1} + \mathbf{L}^2\phi \\ & \quad - \mathbf{L}^1\mathbf{S}^2(T_1, T_2)\phi + \mathbf{L}^1\phi \\ &= (I) + (II) + (III).\end{aligned}\tag{3.13}$$

By (vi) in Lemma 3.7, the terms (I) and (II) in (3.13) converge to 0 when $T_1 \uparrow T_2$. By (i) in Lemma 3.7, the term (III) in (3.13) also converges to 0 when $T_1 \uparrow T_2$. Therefore the scheme is consistent. ■

4 Application to Counterparty Risk of Derivatives

In this section, we apply our multidimensional non-traded assets model (in particular, the one factor model of Section 3) to consider the counterparty risk of derivatives. Our concern as the buyer or holder of the option is that the writer or counterparty may default on the option with payoff $h(S_*)$ at maturity T and we will not receive the full payoff. We have in mind several examples. A natural example is that of a commodity producer who is writing options as part of a hedging program (eg. collars). Some of these options may be written on illiquidly traded assets and thus the option holder is subject to basis risk and in addition, is concerned with the default risk of the option writer. A second example is the default risk of a financial institution who has sold options on various underlying assets - stocks, foreign exchange or commodities. In addition to the possibility of basis risk, the buyer of these options does not always have the ability to trade the underlying asset, or perhaps they choose not to (they may be using the derivative as part of a hedge already). A further example may be that of a purchaser of insurance concerned with the default risk of the insurer. Typically the insured party does not trade at all, which motivates our consideration of this special case. Finally, the option holder may be an employee of a company who receives employee stock options if the company remains solvent. She is restricted from trading the stock of the company, but can trade other indices or stocks in the market. In contrast to the other examples, here the assets of the counterparty and the underlying stock are those of the same company.

We consider an option written on an underlying asset with price S^1 with payoff $h(S_T^1)$ at maturity T . Counterparty default is modeled by comparing the

value of the counterparty's assets S^2 to a default threshold D at maturity, which depends on the liabilities of the counterparty. Following Klein [29] we consider the situation $D = L$, where L refers to the option writer's liabilities, assumed to be a constant.⁴ If the writer defaults, the holder will receive the proportion $h(S_T^1)/L$ of the assets S^2 that her option represents of the writer's liabilities, scaled to reflect a proportional deadweight loss of $\alpha \in [0, 1]$. The payoff of the *vulnerable* option taking counterparty default into account is

$$g(S_T^1, S_T^2) = h(S_T^1)I_{\{S_T^2 \geq L\}} + (1 - \alpha)\frac{h(S_T^1)}{L}S_T^2I_{\{S_T^2 < L\}}. \quad (4.1)$$

To guarantee g is bounded, a sufficient condition is that the payoff $h(S_T^1)$ is bounded, for example, a put option. Note that there is a singular point of g at $S_T^2 = L$, so we have to approximate g by a sequence of (nondecreasing) Lipschitz continuous functions g^ϵ . For the numerical simulation, we only need to choose one g^ϵ for ϵ small enough.

The underlying asset S^1 and the value of the counterparty's assets S^2 are both taken to be non-traded assets so $n = 2$ and prices follow (3.2). The option holder faces some unhedgeable price risk (due to S^1) and some unhedgeable counterparty default risk (due to S^2). She can partially hedge risks by trading the financial index P following (3.1).

Our benchmark models for comparison are those of Johnson and Stulz [28], Klein [29] and Klein and Inglis [30] who take a structural approach to price vulnerable options in a complete market setting and obtain two dimensional Black Scholes style formulas. Other treatments of vulnerable options include complete market models with intertemporal default (see Liang and Ren [38] and the references therein). Implicit in this prior literature are the twin assumptions that the asset underlying the option and the assets of the counterparty can be traded, and therefore, can be used to hedge the counterparty risk of derivatives. Our use of the utility indifference approach is motivated by its recent use in credit risk modeling where the concern is the default of the reference name rather than the default of the counterparty.⁵ Several authors have applied it in modeling of defaultable bonds where the problem remains one dimensional, see in particular Leung et al [34] and Liang and Jiang [35] and Jaimungal and Sigloch [25]. In contrast, options subject to counterparty risk are a natural situation where two or more dimensions arise.

We were also motivated to study indifference pricing of derivatives subject to counterparty risk by the work of Hung and Liu [24] and Murgoci [43]. These papers take a *good deal bounds* approach to pricing, acknowledging the incompleteness of the market but producing prices which are linear in quantity. Furthermore, the method does not allow for any partial hedging on the part of the investor and can produce bounds which can be quite wide.

Based on Theorem 3.8, we give the following approximation scheme for the unit utility indifference price $\mathfrak{C}(s_1, s_2, 0)$ of the vulnerable option. Following (3.8) we make the change of variable $x = \ln s_1$ and $y = \ln s_2$, and define a new

⁴Generalizations to $D = f(S_T^1)$ are easily incorporated and allow for the option liability itself to influence default, eg. $f(x) = h(x) + L$ was considered by Klein and Inglis [30] in a risk neutral setting.

⁵ Utility based pricing has also been utilized by Bielecki and Jeanblanc [7], Sircar and Zariphopolou [48] and recently Jiao et al [26] [27] in an intensity based setting.

operator:

$$\frac{\partial}{\partial \eta} = \bar{\sigma}_1 \frac{\partial}{\partial x} + \bar{\sigma}_2 \frac{\partial}{\partial y}.$$

- (i) Partition $[0, T]$ into N equal intervals:

$$0 = t_0 < t_1 < \dots < t_N = T.$$

- (ii) On $[t_{N-1}, t_N]$, predict the solution by solving the following PDE with the given terminal data g^ϵ :

$$\begin{cases} \partial_t \mathfrak{C}^1 + \frac{1}{2} \partial_{\eta\eta} \mathfrak{C}^1 - \frac{\gamma}{2} (1 - \bar{\kappa}^P) (\partial_\eta \mathfrak{C}^1)^2 = 0, \\ \mathfrak{C}^1|_{t=t_N} = g^\epsilon. \end{cases}$$

The above equation can be linearized via the Cole-Hopf transformation:

$$\bar{\mathfrak{C}}^1 = \exp(-\gamma(1 - \bar{\kappa}^P) \mathfrak{C}^1).$$

In particular, we obtain $\mathfrak{C}^1|_{t=t_{N-1}}$.

- (iii) On $[t_{N-1}, t_N]$, correct the solution by solving the following PDE with the terminal data $\mathfrak{C}^1|_{t=t_{N-1}}$:

$$\begin{cases} \partial_t \mathfrak{C}^2 + \frac{1}{2} \sigma_1^2 \partial_{xx} \mathfrak{C}^2 + \frac{1}{2} \sigma_2^2 \partial_{yy} \mathfrak{C}^2 + A_1 \partial_x \mathfrak{C}^2 + A_2 \partial_y \mathfrak{C}^2 \\ \quad - \frac{\gamma}{2} \sigma_1^2 (\partial_x \mathfrak{C}^2)^2 - \frac{\gamma}{2} \sigma_2^2 (\partial_y \mathfrak{C}^2)^2 = 0, \\ \mathfrak{C}^2|_{t=t_N} = \mathfrak{C}^1|_{t=t_{N-1}} \end{cases}$$

where A_1, A_2 are given in (3.10) to be $A_i = \mu_i - \frac{1}{2}(\sigma_i^2 + \bar{\sigma}_i^2) - \bar{\nu}^P \bar{\sigma}_i$; $i = 1, 2$. The above equation can also be linearized by making the exponential transformation:

$$\bar{\mathfrak{C}}^2 = \exp(-\gamma \mathfrak{C}^2).$$

In particular, we obtain $\mathfrak{C}^2|_{t=t_{N-1}}$, which is used as the approximation of $\mathfrak{C}|_{t=t_{N-1}}$.

- (iv) Repeat the above procedure on $[t_{N-2}, t_{N-1}]$, and obtain $\mathfrak{C}|_{t=t_{N-2}} \dots$

We present results for the European put with payoff $h(S_T^1) = (K - S_T^1)^+$. If S^1 and S^2 are positively correlated, this means when the put option is valuable (in-the-money), the firm's assets S^2 tend to be small, so there is a high risk of default. It is important to take counterparty risk into account for puts in this case, as it will have a relatively large impact on the price. (This would be even more significant when the default trigger involves the option liability). However, for a call, when the call is in-the-money, there is little default risk, and so counterparty risk is less important. Unless otherwise stated, the parameters are: $K = 150$; $T = 1$; $S^1 = 50$; $S^2 = 100$; $L = 1000$; $\alpha = 0.05$; $\gamma = 1$; $\mu_P = 0.1$; $\sigma_P = 0.15$; $\bar{\sigma}_P = 0.2$; $\mu_1 = 0.15$; $\sigma_1 = 0.25$; $\bar{\sigma}_1 = 0.3$; $\mu_2 = 0.1$; $\sigma_2 = 0.3$; $\bar{\sigma}_2 = 0.2$. These parameters result in correlation between the underlying asset and firm's assets of $\rho_{12} = 0.4$; and correlations between each asset and the financial index P of $\rho_{1P} = 0.6$; $\rho_{2P} = 0.4$.

In Figure 1 we show how the approximation converges as we increase the number of time steps N . For our parameter values, $N = 11$ steps is sufficient for the prices to converge and we use it in all subsequent figures. We aim to compare the utility indifference price with hedging in the *financial index* with the benchmark risk neutral price in a complete market (computed as in Lemma 3.3 with $n = 2$, as studied in Klein [29]). We also compare to the situation where the financial index is independent of the other assets and thus there is *no hedging* carried out. This price was given earlier in Corollary 3.2. Figure 2 provides a demonstration of the accuracy of the algorithm. We take $\bar{\sigma}_P = 0$ and compare the splitting approximation to the formula in Corollary 3.2.

Figure 3 shows the vulnerable option price(s) against the underlying asset price S^1 . The two panels of Figure 3 are intended to illustrate a “close or likely to default” scenario (the left panel with $S^2 = 500$ relative to $L = 1000$) and a “far or unlikely to default” scenario (the right panel with $S^2 = 1400$ relative to $L = 1000$). As expected, in both panels the risk neutral or complete market price is the highest. In both panels, as the underlying asset price becomes very large, all option prices tend to zero, as the put is worthless, regardless of the default. At $S^1 = 0$, in the right panel, the option price is equal to the option strike $K = 150$. In the left panel, the option price is lower due to the risk of counterparty default. As S^1 increases, all option prices decrease, as the moneyness of the put changes. When S^1 is close to zero, we see a dramatic drop in the utility indifference prices (relative to the risk neutral prices) due to the risk aversion towards unhedgeable price and default risks. Recall that since the underlying asset and firm’s assets are positively correlated, default risk becomes more important for low values of the underlying asset. The price drop is much more significant in the left panel (and in this extreme case, the option price drops down to zero if no hedging can be carried out), where the likelihood of counterparty default is higher. The option holder’s risk aversion causes the utility indifference prices to lie below the risk neutral price (in each default scenario) with the relative discount to the risk neutral price being much greater in the left panel where default is more likely. Assuming the holder can hedge in the financial index, there is a drop of around 75% from the risk-neutral price to the utility indifference price. In the right panel, where the likelihood of default is relatively low, the difference between the utility indifference price(s) and the risk-neutral price is not as dramatic, and is at most around 20% of the risk-neutral price. We also see that the ability to hedge in the correlated financial index (versus no hedging at all) is more important when the default risk is higher (in the left panel).

Figure 4 displays the impact of the option writer’s asset value S^2 on the option price for a fixed asset price S^1 . We see a dramatic difference in the behavior of the risk neutral price and the utility indifference prices. Under risk neutrality, the option price increases smoothly with S^2 . However, under utility indifference, the prices are low and do not change much with values of S^2 below the default trigger of $L = 1000$. This is despite the put being in-the-money. As S^2 increases beyond the default level, the likelihood of default diminishes, and the utility indifference prices start to increase with S^2 . Note that the utility price is not always below the risk neutral price. Although Proposition 3.6 tells us that the risk neutral price is obtained as a limiting case of the utility indifference price, it requires condition (3.7) to hold.

Figure 5 compares how the vulnerable option price changes with risk aver-

sion parameter γ , and the idiosyncratic volatilities σ_1, σ_2 . The left panel plots vulnerable option prices against maturity T for various values of risk aversion γ . We see that the more risk averse the option holder is, the less she will pay for the option, consistent with Proposition 3.4. The other observation is that option prices for a fixed γ are decreasing with maturity T . The risk neutral price is also decreasing with T , albeit very gradually. This is in contrast to risk neutral prices for non-default European put options which will increase in T provided there are no dividends. The reason is that there is a tradeoff between price and default risk. If the maturity is longer, there is more chance for both S^1 and S^2 to fall - S^1 falling means the put is more valuable, but S^2 falling increases the default risk. For the parameters considered, the default risk is the dominant factor and thus the option price decreases with T . This is also in contrast to the call option, where Klein [29] reports that the risk neutral price increases with maturity.

Recall that we do not expect price monotonicity in terms of the correlations, except in the situation outlined in Proposition 3.5. Here we give an example of prices for various values of the idiosyncratic volatilities σ_1, σ_2 . The left panel sets parameters to be $\mu_1 = 0.1$ and $\mu_2 = 0.06$ to satisfy the CAPM restriction on Sharpe ratios. If $\sigma_1 = \sigma_2 = 0$, then we have $\rho_{12} = 1$, $\rho_{1P} = 0.8$ and $\rho_{2P} = 0.8$. Similarly if $\sigma_1 = 0.25$ and $\sigma_2 = 0.3$ then $\rho_{12} = 0.4$, $\rho_{1P} = 0.6$ and $\rho_{2P} = 0.4$. Finally, if $\sigma_1 = \sigma_2 = 1$ then $\rho_{12} = 0.06$, $\rho_{1P} = 0.2$ and $\rho_{2P} = 0.16$. We see that as σ_1, σ_2 increase, the utility indifference price falls. Correspondingly, as the correlations $\rho_{12}, \rho_{1P}, \rho_{2P}$ increase, the option price rises.

5 Concluding Remarks

We close with several suggestions for further work. In this paper, we prove the convergence of our splitting method for the utility indifference pricing PDE but do not concern ourselves with the convergence rate. Tools such as Krylov's idea of "shaking the coefficients" (see [33]) and Barles and Jakobsen's "switching system" (see [3]) could be useful. A second suggestion for further investigation is to employ the splitting method to give a numerical scheme to solve quadratic BSDEs. To achieve this goal, the first problem may be to understand the *monotone scheme* from a probabilistic perspective, we refer to Cont and Voltchkova [9] and Dai and Wu [12].

Several developments in the Markovian setting of Section 3 may be of interest. A first comment is that the splitting relies on the special setup of the one factor model and it does not seem possible to extend beyond this case. It would be interesting to consider alternative numerical methods which could be used in the general multidimensional framework. Second, following Eberlein and Madan [15] we could extend our counterparty risk model in Section 4 to treat assets and liabilities as separate stochastic processes. Finally, in this paper we price European claims in a multidimensional non-traded assets framework. Extensions to corresponding American claims in a multidimensional model (following work of Oberman and Zariphopoulou [46] and Henderson [20] in one dimension) would allow us to consider intertemporal counterparty default.

Appendix

Proof of Lemma 2.2. We characterize the value function of the optimization problem (2.4) by the solution of quadratic BSDE. By plugging (2.3) into (2.4), we have

$$\begin{aligned} & \sup_{\pi \in \mathcal{A}_{ad}[0, T]} E^{\mathbf{P}} \left[-e^{-\gamma(x - \mathfrak{C}_0^\lambda + \int_0^T \pi_s (\mu_s^P ds + \langle \sigma_s^P, d\mathcal{W}_s \rangle) + \lambda g(\mathcal{S}.)} \right] \\ &= -e^{-\gamma(x - \mathfrak{C}_0^\lambda)} \inf_{\pi \in \mathcal{A}_{ad}[0, T]} E^{\mathbf{P}} \left[e^{-\gamma(\int_0^T \pi_s (\mu_s^P ds + \langle \sigma_s^P, d\mathcal{W}_s \rangle) + \lambda g(\mathcal{S}.)} \right] \\ &= -e^{-\gamma(x - \mathfrak{C}_0^\lambda)} \exp \left\{ -\gamma \sup_{\pi \in \mathcal{A}_{ad}[0, T]} \mathcal{Y}_0(\pi) \right\}, \end{aligned}$$

where $\mathcal{Y}_0(\pi)$ denotes the risk-sensitive control criterion:

$$\mathcal{Y}_0(\pi) = \frac{-1}{\gamma} \ln E^{\mathbf{P}} \left[e^{-\gamma(\int_0^T \pi_s (\mu_s^P ds + \langle \sigma_s^P, d\mathcal{W}_s \rangle) + \lambda g(\mathcal{S}.)} \right].$$

We further introduce the risk-sensitive control problem:

$$\mathcal{Y}_0 = \sup_{\pi \in \mathcal{A}_{ad}[0, T]} \mathcal{Y}_0(\pi).$$

In the following, we characterize both $\mathcal{Y}_0(\pi)$ and \mathcal{Y}_0 by the solutions of quadratic BSDEs.

Let us first consider the risk-sensitive control criterion $\mathcal{Y}_0(\pi)$. For any given trading strategy $\pi \in \mathcal{A}_{ad}[0, T]$, we define \mathbf{P} -*BMO* martingale:

$$N_t = - \int_0^t \gamma \pi_s \langle \sigma_s^P, d\mathcal{W}_s \rangle, \quad \text{for } t \in [0, T].$$

Indeed, for any \mathcal{F}_t -stopping time $\tau \in [0, T]$, we have

$$\begin{aligned} \sup_{\tau} E^{\mathbf{P}} [|N_T - N_\tau|^2 | \mathcal{F}_\tau] &= \sup_{\tau} E^{\mathbf{P}} \left[\int_\tau^T \gamma^2 \|\sigma_s^P\|^2 |\pi_s|^2 ds \middle| \mathcal{F}_\tau \right] \\ &\leq CT \sup_{\tau} E^{\mathbf{P}} \left[\int_\tau^T |\pi_s|^2 ds \middle| \mathcal{F}_\tau \right] < \infty. \end{aligned}$$

Hence the Doléans-Dade exponential $\mathcal{E}(N)$ is uniformly integrable, and we change the probability measure by defining $\frac{d\bar{\mathbf{Q}}}{d\mathbf{P}} = \mathcal{E}(N)$. Under the new probability measure $\bar{\mathbf{Q}}$, the risk-sensitive control criterion becomes

$$\mathcal{Y}_0(\pi) = \frac{-1}{\gamma} \ln E^{\bar{\mathbf{Q}}} \left[e^{-\int_0^T (\gamma \mu_s^P \pi_s - \frac{1}{2} \gamma^2 \|\sigma_s^P\|^2 |\pi_s|^2) ds} e^{-\gamma \lambda g(\mathcal{S}.)} \right]$$

which is the unique solution to the following quadratic BSDE:

$$\mathcal{Y}_t(\pi) = \lambda g(\mathcal{S}.) + \int_t^T \left(F_s(\pi) - \frac{\gamma}{2} \|\mathcal{Z}_s\|^2 \right) ds - \int_t^T \langle \mathcal{Z}_s, d\bar{\mathcal{W}}_s \rangle \quad (0.1)$$

with

$$F_s(\pi) = \mu_s^P \pi_s - \frac{\gamma}{2} \|\sigma_s^P\|^2 |\pi_s|^2,$$

and $\bar{\mathcal{W}} = \mathcal{W} - [\mathcal{W}, N]$ being the Brownian motion under $\bar{\mathbf{Q}}$. Indeed, note that (0.1) can be reformulated as

$$\mathcal{Y}_t(\pi) + \int_0^t F_s(\pi) ds = \lambda g(\mathcal{S}_\cdot) + \int_0^T F_s(\pi) ds - \int_t^T \frac{\gamma}{2} \|\mathcal{Z}_s\|^2 ds - \int_t^T \langle \mathcal{Z}_s, d\bar{\mathcal{W}}_s \rangle.$$

By changing variables:

$$\bar{\mathcal{Y}}_t(\pi) = e^{-\gamma(\mathcal{Y}_t(\pi) + \int_0^t F_s(\pi) ds)}; \quad \bar{\mathcal{Z}}_t = -\gamma \bar{\mathcal{Y}}_t(\pi) \mathcal{Z}_t,$$

we have

$$\bar{\mathcal{Y}}_t(\pi) = e^{-\gamma(\lambda g(\mathcal{S}_\cdot) + \int_0^T F_s(\pi) ds)} - \int_t^T \langle \bar{\mathcal{Z}}_s, d\bar{\mathcal{W}}_s \rangle,$$

which has the explicit solution:

$$\begin{aligned} \bar{\mathcal{Y}}_0(\pi) &= E^{\bar{\mathbf{Q}}} \left[e^{-\gamma(\lambda g(\mathcal{S}_\cdot) + \int_0^T F_s(\pi) ds)} \right] \\ &= E^{\bar{\mathbf{Q}}} \left[e^{-\int_0^T (\gamma \mu_s^P \pi_s - \frac{1}{2} \gamma^2 \|\sigma_s^P\|^2 |\pi_s|^2) ds} e^{-\gamma \lambda g(\mathcal{S}_\cdot)} \right]. \end{aligned}$$

Next, we consider the risk-sensitive control problem for \mathcal{Y}_0 . Note that under the probability measure \mathbf{P} , (0.1) becomes

$$\mathcal{Y}_t(\pi) = \lambda g(\mathcal{S}_\cdot) + \int_t^T \left(F_s(\pi) - \gamma \langle \sigma_s^P, \mathcal{Z}_s \rangle \pi_s - \frac{\gamma}{2} \|\mathcal{Z}_s\|^2 \right) ds - \int_t^T \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle. \quad (0.2)$$

By the comparison principle for quadratic BSDE, $\mathcal{Y}_0 = \sup_{\pi \in \mathcal{A}_{ad}[0, T]} \mathcal{Y}_0(\pi)$ is the unique solution to the following quadratic BSDE:

$$\mathcal{Y}_t = \lambda g(\mathcal{S}_\cdot) + \int_0^T F_s ds - \int_t^T \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle$$

with

$$\begin{aligned} F_s &= \sup_{\pi_s} \left\{ F_s(\pi) - \gamma \langle \sigma_s^P, \mathcal{Z}_s \rangle \pi_s - \frac{\gamma}{2} \|\mathcal{Z}_s\|^2 \right\} \\ &= -\frac{\gamma}{2} \|\mathcal{Z}_s\|^2 + \frac{\gamma}{2 \|\sigma_s^P\|^2} \left| \langle \sigma_s^P, \mathcal{Z}_s \rangle - \frac{\mu_s^P}{\gamma} \right|^2, \end{aligned}$$

and

$$\bar{\pi}_s^* = -\frac{\langle \sigma_s^P, \mathcal{Z}_s \rangle}{\|\sigma_s^P\|^2} + \frac{\mu_s^P}{\gamma \|\sigma_s^P\|^2}.$$

To verify that $\bar{\pi}^* \in \mathcal{A}_{ad}[0, T]$, we only need to note that $\int_0^\cdot \langle \mathcal{Z}_s, d\mathcal{W}_s \rangle$ is a \mathbf{P} -BMO martingale and Assumptions (A1) and (A2) on the coefficients.

The optimization problem (2.5) is a special case of (2.4) with $\lambda = 0$, whose solution is

$$-\exp \left\{ -\gamma \left(x + \frac{1}{2\gamma} \int_0^T \frac{|\mu_s^P|^2}{\|\sigma_s^P\|^2} ds \right) \right\},$$

and the optimal trading strategy π^* is given by $\pi_t^* = \frac{\mu_t^P}{\gamma \|\sigma_t^P\|^2}$.

Finally, by Definition 2.1, the price \mathfrak{C}_0^λ is given by the solution to

$$\begin{aligned} -e^{-\gamma(x - \mathfrak{C}_0^\lambda + \mathcal{Y}_0)} &= E^{\mathbf{P}} \left[-e^{-\gamma \left(X_T^{x - \mathfrak{C}_0^\lambda}(\bar{\pi}^*) + \lambda g(\mathcal{S}_\cdot) \right)} \right] \\ &= E^{\mathbf{P}} \left[-e^{-\gamma X_T^x(\pi^*)} \right] = -e^{-\gamma \left(x + \frac{1}{2\gamma} \int_0^T \frac{|\mu_s^P|^2}{\|\sigma_s^P\|^2} ds \right)}, \end{aligned}$$

Therefore,

$$\begin{aligned} \mathfrak{C}_0^\lambda &= \mathcal{Y}_0 - \frac{1}{2\gamma} \int_0^T \frac{|\mu_s^P|^2}{\|\sigma_s^P\|^2} ds \\ &= \lambda g(\mathcal{S}_\cdot) + \int_0^T F_s ds - \int_0^T \mathcal{Z}_s \cdot d\mathcal{W}_s - \frac{1}{2\gamma} \int_0^T \frac{|\mu_s^P|^2}{\|\sigma_s^P\|^2} ds \\ &= \lambda g(\mathcal{S}_\cdot) + \int_0^T f(s, \mathcal{Z}_s) ds - \int_0^T \mathcal{Z}_s \cdot d\mathcal{W}_s, \end{aligned}$$

and the hedging strategy for such λ units of the option is given by

$$\bar{\pi}_t^* - \pi_t^* = -\frac{\langle \sigma_t^P, \mathcal{Z}_t \rangle}{\|\sigma_t^P\|^2} + \frac{\mu_t^P}{\gamma \|\sigma_t^P\|^2} - \frac{\mu_t^P}{\gamma \|\sigma_t^P\|^2} = -\frac{\langle \sigma_t^P, \mathcal{Z}_t \rangle}{\|\sigma_t^P\|^2},$$

which completes the proof.

Proof of Theorem 2.5. The type of FBSDE is in fact a special case of functional differential equations studied by Liang et al in [36] and [37]. We define $\mathfrak{S}([0, T], \mathbb{R}^n)$, the space of continuous and \mathcal{F}_t -adapted processes valued in \mathbb{R}^n such that $\sup_{t \in [0, T]} \sum_{i=1}^n |S_t^i| \in L^2(\Omega, \mathcal{F}_T, \mathbf{Q})$ and endowed with the norm:

$$\|\mathcal{S}\|_{\mathfrak{S}} = E^{\mathbf{Q}} \left\{ \sup_{t \in [0, T]} \sum_{i=1}^n |S_t^i|^2 \right\}^{1/2}.$$

Then for \mathcal{S}^m and \mathcal{S}^{m+1} , we have

$$\begin{aligned} &E^{\mathbf{Q}} \left\{ \sup_{t \in [0, T]} \sum_{i=1}^n |\ln S_t^{m+1, i} - \ln S_t^{m, i}|^2 \right\}^{1/2} \\ &= E^{\mathbf{Q}} \left\{ \sup_{t \in [0, T]} \sum_{i=1}^n \left| \int_0^t \frac{\gamma}{2} \langle \sigma_s^i, \mathcal{Z}_s^m - \mathcal{Z}_s^{m-1} \rangle - \frac{\gamma \langle \sigma_s^i, \sigma_s^P \rangle}{2 \|\sigma_s^P\|} \langle \sigma_s^P, \mathcal{Z}_s^m - \mathcal{Z}_s^{m-1} \rangle ds \right|^2 \right\}^{1/2} \\ &\leq C\sqrt{T} E^{\mathbf{Q}} \left\{ \int_0^T \|\mathcal{Z}_s^m - \mathcal{Z}_s^{m-1}\|^2 ds \right\}^{1/2} \\ &= C\sqrt{T} E^{\mathbf{Q}} \left\{ \left| \int_0^T \langle \mathcal{Z}^m - \mathcal{Z}^{m-1}, d\mathcal{B}_s \rangle \right|^2 \right\}^{1/2} \\ &\leq C\sqrt{T} E^{\mathbf{Q}} \{ |g(\mathcal{S}^m) - g(\mathcal{S}^{m-1})|^2 \}^{1/2} \\ &\leq CK\sqrt{T} E^{\mathbf{Q}} \left\{ \sup_{t \in [0, T]} \sum_{i=1}^n |\ln S_t^{m, i} - \ln S_t^{m-1, i}|^2 \right\}^{1/2}. \end{aligned}$$

We iterate the above inequality and obtain

$$\|\ln \mathcal{S}^{m+1} - \ln \mathcal{S}^m\|_{\mathfrak{S}} \leq (CK\sqrt{T})^m \|\ln \mathcal{S}^1 - \ln \mathcal{S}^0\|_{\mathfrak{S}}.$$

Hence if either T or K is small enough, for any natural number p ,

$$\begin{aligned} \|\ln \mathcal{S}^{m+p} - \ln \mathcal{S}^m\|_{\mathfrak{S}} &\leq \sum_{j=1}^p \|\ln \mathcal{S}^{m+j} - \ln \mathcal{S}^{m+j-1}\|_{\mathfrak{S}} \\ &\leq \frac{(CK\sqrt{T})^m}{1 - CK\sqrt{T}} \|\ln \mathcal{S}^1 - \ln \mathcal{S}^0\|_{\mathfrak{S}} \rightarrow 0 \end{aligned}$$

when $m \rightarrow \infty$. Therefore, $\ln \mathcal{S}^m$ is a Cauchy sequence in $\mathfrak{S}([0, T]; \mathbb{R}^n)$ and converges to some $\ln \mathcal{S}$, and moreover, by Dominated Convergence Theorem,

$$\lim_{m \rightarrow \infty} E^{\mathbf{Q}}[\lambda g(\mathcal{S}^m)] = E^{\mathbf{Q}}[\lambda g(\mathcal{S})].$$

The rest of the proof is to verify $E^{\mathbf{Q}}[\lambda g(\mathcal{S})] = \mathfrak{C}_0^\lambda$, which follows by a reverse use of Girsanov's transformation in Theorem 2.4, and we leave it to the reader.

References

- [1] Ankirchner, S., Imkeller, P., and G. Reis, Pricing and hedging of derivatives based on non- tradable underlyings, *Mathematical Finance* 20(2) (2010) 289–312.
- [2] Barles G., The convergence of approximation schemes for parabolic equations arising in finance theory, *Numerical methods in finance*, Cambridge University Press (1997) 1–21.
- [3] Barles, G. and E. R. Jakobsen, Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations, *Mathematics of Computation* 76 (2007) 1861–1893.
- [4] Barles, G. and P. E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Asymptotic Analysis* 4(3) (1991) 271–283.
- [5] Becherer D., Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging, *Ann. Appl. Probab.* 16 (2006) 2027–2054.
- [6] Bielecki T.R., Crepey S., Jeanblanc M. and B. Zagari, Valuation and hedging of CDS counterparty exposure in a markov copula model, *International Journal of Theoretical and Applied Finance*, to appear.
- [7] Bielecki, T. R. and M. Jeanblanc, Indifference pricing of defaultable claims, *Indifference Pricing*, edited by R. Carmona, Princeton University Press (2009) 211–240.
- [8] Brigo D. and Chourdakis K., Counterparty risk for credit default swaps: Impact of spread volatility and default correlation, *International Journal of Theoretical and Applied Finance* 12 (2009) 1007–1026.
- [9] Cont, R. and E. Voltchkova, A finite difference scheme for option pricing in jump diffusion and exponential Levy models, *SIAM Journal on Numerical Analysis*, 4 (2005) 1596–1626.
- [10] Carmona, R. (editor), Indifference pricing, theory and applications, *Princeton University Press* (2009).
- [11] Carmona, R. and M. Ludkovski, Pricing commodity derivatives with basis risk and partial observations, *Preprint* (2006).
- [12] Dai, M. and L. Wu, Pricing jump risk with utility indifference, *Quantitative Finance*, 9(2) (2009) 177–186.
- [13] Davis, M. H. A., Optimal hedging with basis risk, *Second Bachelier Colloquium on Stochastic Calculus and Probability*, edited by Y. Kabanov, R. Lipster, and J. Stoyanov, Springer (2006) 169–187.
- [14] Delarue, F., Estimates of the solutions of a system of quasi-linear PDEs: a probabilistic scheme, *Séminaire de Probabilités XXXVII, Lecture Notes in Math., Springer*, 1832, 2003, 290–332.
- [15] Eberlein, E. and D. Madan, Unlimited liabilities, reserve capital requirements and the taxpayer put option, *Working Paper* (2009).
- [16] Frei C. and M. Schweizer, Exponential utility indifference valuation in two Brownian settings with stochastic correlation, *Advances in Applied Probability* 40 (2008) 401–423.
- [17] Frei, C. and Schweizer, M., Exponential Utility Indifference Valuation in a General Semimartingale Model, in: *F. Delbaen, M. Rásonyi and C. Stricker (eds.), Optimality and Risk: Modern Trends in Mathematical Finance*, Springer, (2009), 49–86.
- [18] Henderson, V., Valuation of claims on nontraded assets using utility maximization, *Mathematical Finance* 12 (2002) 351–373.

- [19] Henderson V., The impact of the market portfolio on the valuation, incentives and optimality of executive stock options, *Quantitative Finance*, 5(1) (2005) 1–13.
- [20] Henderson V., Valuing the option to invest in an incomplete market, *Mathematics and Financial Economics* 1 (2007) 103–128.
- [21] Henderson, V. and D. Hobson, Substitute hedging, *RISK* 15(5) (2002) 71–75. (Reprinted in *Exotic Options: The Cutting Edge Collection*, RISK Books, London, 2003.)
- [22] Henderson, V. and D. Hobson, Utility indifference pricing: an overview, *Indifference Pricing*, edited by R. Carmona, Princeton University Press (2009) 44–74.
- [23] Hu, Y., P. Imkeller and M. Müller, Utility maximization in incomplete markets, *Annals of Applied Probability* 15 (2005) 1691–1712.
- [24] Hung, M. and Y. Liu, Pricing vulnerable options in incomplete markets, *Journal of Futures markets* 25 (2005) 135–170.
- [25] Jaimungal, S. and G. Sigloch, Incorporating risk and ambiguity aversion into a hybrid model of default, *Mathematical Finance*, to appear.
- [26] Jiao, Y. and H. Pham, Optimal investment with counterparty risk: a default-density modeling approach, *Finance and Stochastics*, to appear.
- [27] Jiao, Y., I. Kharroubi and H. Pham, Optimal investment under multiple defaults risk: a BSDE-decomposition approach, *Annals of Applied Probability*, to appear.
- [28] Johnson, H. and R. Stulz, The pricing of options with default risk, *Journal of Finance* 42 (1987) 267–280.
- [29] Klein, P., Pricing Black-Scholes options with correlated credit risk, *Journal of Banking and Finance* 20 (1996) 1211–1229.
- [30] Klein, P. and M. Inglis, Pricing vulnerable European options when the option’s payoff can increase the risk of financial distress, *Journal of Banking and Finance* 25 (2001) 993–1012.
- [31] Kobylanski, M., Backward stochastic differential equations and partial differential equations with quadratic growth, *Annals of Probability* 28 (2000) 558–602.
- [32] Kramkov, D. and M. Sirbu, Asymptotic analysis of utility-based hedging strategies for small number of contingent claims, *Stochastic Processes and Their Applications*, 117(11) (2007) 1606–1620.
- [33] Krylov, N. V., On the rate of convergence of finite-difference approximations for Bellman’s equations with variable coefficients, *Probability Theory and Related Fields* 117(1) (2000) 1–16.
- [34] Leung, T., S. Sircar and T. Zariphopoulou, Credit derivatives and risk aversion, *Advances in Econometrics*, edited by T. Fomby, J.-P. Fouque and K. Solna, Elsevier Science 22 (2008) 275–291.
- [35] Liang, G. and L. Jiang, A modified structural model for credit risk, *IMA Journal of Management Mathematics*, to appear.
- [36] Liang, G., T. Lyons and Z. Qian, Backward stochastic dynamics on a filtered probability space, *Annals of Probability*, 39(4) (2011) 1422–1448.
- [37] Liang, G., T. Lyons and Z. Qian, A functional approach to FBSDEs and its application in optimal portfolios, *Working paper* (2010).
- [38] Liang, G. and X. Ren, The credit risk and pricing of OTC options, *Asia-Pacific Financial Markets* 14(1) (2007) 45–68.
- [39] Mania M. and M. Schweizer, Dynamic exponential utility indifference valuation, *Ann. Appl. Probab.* 15 (2005) 2113–2143

- [40] Marchuk, G. I., Some application of splitting-up methods to the solution of mathematical physics problems, *Apl. Mat.* 13 (1968) 103–132.
- [41] Marchuk G.I., Splitting and alternating direction methods, *Handbook of Numerical Analysis* 35 (1990) 435–461.
- [42] Morlais, M. A., Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem, *Finance and Stochastics*, 13(1), (2009), 121–150.
- [43] Murgoci, A., Vulnerable options and good deal bounds: a structural model, *Working paper* (2008).
- [44] Musiela, M. and T. Zariphopoulou, An example of indifference prices under exponential preferences, *Finance and Stochastics* 8 (2004) 229–239.
- [45] Nadtochiy S. and T. Zariphopoulou, An approximation scheme for the solution to the optimal investment problem in incomplete markets, *Working paper* (2011).
- [46] Oberman, A. and T. Zariphopoulou, Pricing early exercise contracts in incomplete markets, *Computational Management Science* 1 (2003) 75–107.
- [47] Sircar, R. and T. Zariphopoulou, Bounds and asymptotic approximations for utility prices when volatility is random, *SIAM Journal on Control and Optimization* 43(4) (2005) 1328–1353.
- [48] Sircar, R. and T. Zariphopoulou, Utility valuation of multi-name credit derivatives and application to CDOs, *Quantitative Finance* 10(2) (2010) 195–208.
- [49] Tehranchi M., Explicit solutions of some utility maximization problems in incomplete markets, *Stochastic Processes and Their Applications* 114(1) (2004) 109–125.
- [50] Tourin, A., Splitting methods for Hamilton-Jacobi equations, *Numer. Methods Partial Differential Equations* 22(2) (2006) 381–396.

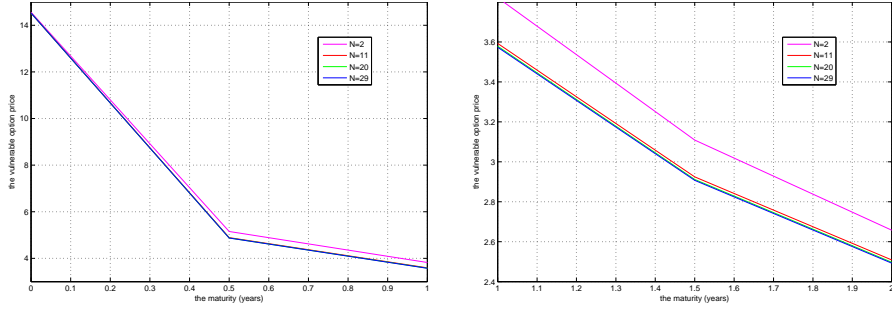


Figure 1: Approximation of the option price for various time steps N . The left panel takes $T \in [0, 1]$; the right panel takes $T \in [1, 2]$.

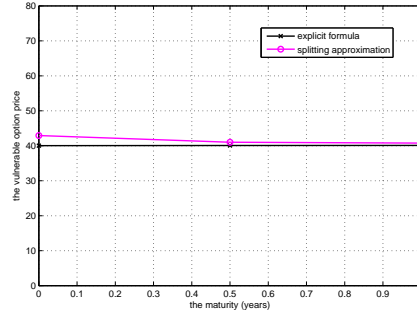


Figure 2: Comparison of the explicit price and the approximated price via splitting when $\bar{\sigma}^P = 0$, $S^2 = 1400$, and $N = 11$.

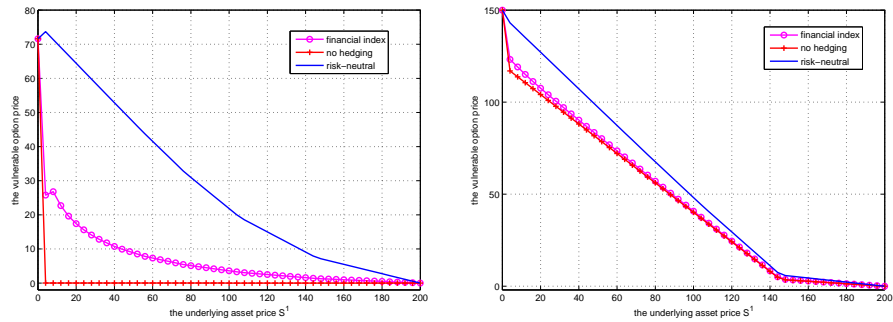


Figure 3: Vulnerable option price against the underlying asset price S^1 . The left panel takes $S^2 = 500$; the right panel takes $S^2 = 1400$.

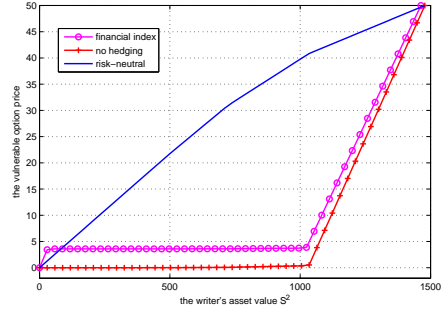


Figure 4: Vulnerable option price against the writer's asset value S^2 .

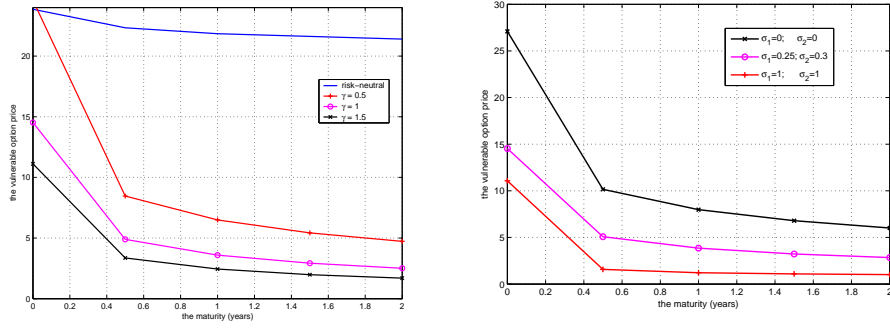


Figure 5: Impact of risk aversion and correlation. The left panel gives the option price against maturity for various risk aversion parameters γ . The right panel gives the price against various correlation parameters. We set $\mu_1 = 0.1$ and $\mu_2 = 0.06$ to satisfy the parameter restriction of Proposition 3.5.